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Approximation Spaces

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Using the concept of an approximation space we describe certain analogies between spaces of sequences, functions and operators. In order to illustrate the power of this method, some old and new theorems about distributions of Fourier coefficients and eigenvalues are established.

The aim of this paper is to describe some analogies between well-known spaces of sequences, functions and operators which have for the first time been observed by J. Peetre. For this purpose we develop a theory of so-called approximation spaces. Similar concepts were already investigated by P. L. Butzer and K. Scherer, J. A. Brudnij and N. J. Krugljak, as well as by J. Peetre and G. Sparr within the framework of their interpolation theory of abelian groups. For the convenience of the reader and in order to be self-contained, we present some of their basic theorems with new and very simple proofs, and apply them to the main examples.

From our point of view, the most interesting results are to be found in the last chapter, where we treat some significant applications. First, we investigate the influence of smoothness properties of periodic functions on the behaviour of their Fourier coefficients. This is a special case of a more general problem. If we remember that the complex Fourier coefficients coincide with the eigenvalues of the corresponding convolution operators, it suggests itself to ask, how do the distributions of eigenvalues of integral operators depend on certain properties of the generating kernels? Results can be obtained via a generalization of Weyl's theorem which we prove by the method of related operators. The original proof (unpublished) of this remarkable theorem is due to B. Carl and H. König.

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1. Definitions and Elementary Properties

In this chapter we introduce the concept of an approximation space and formulate some elementary properties.

1.1. *Quasi-Banach Spaces*

A *quasi-norm* is a non-negative function $\|\cdot\|_X$ defined on a (real or complex) linear space X for which the following conditions are satisfied:

(1) If $\|f\|_X = 0$ for some $f \in X$, then $f = o$.

(2) $\|\lambda f\|_X = |\lambda| \, \|f\|_X$ for $f \in X$ and all scalars λ.

(3) There exists a constant $c_X \geqslant 1$ such that

$$\|f+g\|_X \leqslant c_X[\|f\|_X + \|g\|_X] \qquad \text{for} \quad f, g \in X.$$

The quasi-norms $\|\cdot\|_X^{(1)}$ and $\|\cdot\|_X^{(2)}$ are said to be *equivalent* if

$$\|f\|_X^{(2)} \leqslant c_1 \|f\|_X^{(1)} \qquad \text{and} \qquad \|f\|_X^{(1)} \leqslant c_2 \|f\|_X^{(2)} \qquad \text{for all } f \in X,$$

where c_1 and c_2 are suitable constants.

Every quasi-norm generates a metrisable Hausdorff topology on the underlying linear space. For two equivalent quasi-norms the corresponding topologies coincide.

A *quasi-Banach space* is a linear space X equipped with a quasi-norm $\|\cdot\|_X$ such that every Cauchy sequence is convergent.

A quasi-norm $\|\cdot\|_X$ is called a *p-norm* $(0 < p \leqslant 1)$ if

$$\|f+g\|_X^p \leqslant \|f\|_X^p + \|g\|_X^p \qquad \text{for} \quad f, g \in X.$$

Then condition (3) is satisfied with $c_X := 2^{1/p-1}$. Conversely, for every quasi-norm $\|\cdot\|_X$ there exists an equivalent p-norm $\|\cdot\|_X^0$, if we determine p by $1/p := 1 + \log_2 c_X$. Clearly, every p-norm is also a q-norm for $0 < q < p \leqslant 1$.

Let X and Y be quasi-Banach spaces. Then $\mathfrak{L}(X, Y)$ denotes the linear space of all (bounded linear) operators T acting from X into Y. Putting

$$\|T\|_{\mathscr{L}} := \sup\{\|Tf\|_Y : \|f\|_X \leqslant 1\}$$

we get a quasi-norm on $\mathfrak{L}(X, Y)$.

1.2. *Approximation Schemes*

An *approximation scheme* (X, A_n) is a quasi-Banach space X together with a sequence of subsets A_n such that the following conditions are satisfied:

(1) $A_1 \subseteq A_2 \subseteq \cdots \subseteq X$.

(2) $\lambda A_n \subseteq A_n$ for all scalars λ and $n = 1, 2, \ldots$.

(3) $A_m + A_n \subseteq A_{m+n}$ for $m, n = 1, 2, \ldots$.

We put $A_0 := \{o\}$. Obviously, $A := \bigcup_{n=1}^{\infty} A_n$ is a linear subset.

1.3. *Approximation Numbers*

Let (X, A_n) be an approximation scheme. For $f \in X$ and $n = 1, 2, \ldots$ the nth *approximation number* is defined by

$$\alpha_n(f, X) := \inf\{\|f - a\|_X : a \in A_{n-1}\}.$$

Now some elementary properties are listed:

(1) $\|f\|_X = \alpha_1(f, X) \geqslant \alpha_2(f, X) \geqslant \cdots \geqslant 0$ for $f \in X$.

(2) $\alpha_n(\lambda f, X) = |\lambda| \, \alpha_n(f, X)$ for $f \in X$, all scalars λ, and $n = 1, 2, \ldots$.

(3) $\alpha_{m+n-1}(f+g, X) \leqslant c_X[\alpha_m(f, X) + \alpha_n(g, X)]$ for $f, g \in X$ and $m, n = 1, 2, \ldots$.

It is worthwhile mentioning that $\alpha_n(f, X)$ is in general not a continuous function of f. This unpleasant situation can be avoided if we use an equivalent p-norm on X.

1.4. *Approximation Spaces*

Let $0 < \rho < \infty$ and $0 < u \leqslant \infty$. Then the *approximation space* X_u^ρ, or more precisely $(X, A_n)_u^\rho$, consists of all elements $f \in X$ such that $(n^{\rho - 1/u} \alpha_n(f, X)) \in l_u$, where $n = 1, 2, \ldots$. We put

$$\|f\|_{X_u^\rho} := \|(n^{\rho - 1/u} \alpha_n(f, X))\|_{l_u} \qquad \text{for} \quad f \in X_u^\rho.$$

The proof of the following result is straightforward.

PROPOSITION 1. *X_u^ρ is a quasi-Banach space.*

Remark. If X is a Banach space and we have $A_n + A_n = A_n$ for $n = 1, 2, \ldots$, then X_u^ρ with $1 \leqslant u \leqslant \infty$ is a Banach space, as well. This case was treated in detail by Butzer and Scherer [6].

Now we formulate a very useful criterion.

PROPOSITION 2. *An element $f \in X$ belongs to X_u^ρ if and only if $(2^{k\rho} \alpha_{2^k}(f, X)) \in l_u$, where $k = 0, 1, \ldots$. Moreover,*

$$\|f\|_{X_u^\rho}^{\exp} := \|(2^{k\rho} \alpha_{2^k}(f, X))\|_{l_u}$$

defines an equivalent quasi-norm on X_u^ρ.

Next we state that the scale of approximation spaces is lexicographically ordered.

PROPOSITION 3. *There holds*

$$X_{u_1}^{\rho_1} \supseteq X_{u_2}^{\rho_2} \quad \textit{for} \quad 0 < \rho_1 < \rho_2 < \infty \textit{ and } 0 < u_1, u_2 \leqslant \infty.$$

and

$$X_{u_1}^{\rho} \subseteq X_{u_2}^{\rho} \quad \textit{for} \quad 0 < \rho < \infty \textit{ and } 0 < u_1 < u_2 \leqslant \infty.$$

Finally, we mention a deep result of Peetre and Sparr [14], stating that the approximation spaces form a real interpolation scale. See also [6, Korollar 2.3.1].

PROPOSITION 4. *Let* $0 < \theta < 1$ *and* $\rho = (1-\theta)\rho_0 + \theta\rho_1$, *where* $\rho_0 \neq \rho_1$. *Then*

$$(X_{u_0}^{\rho_0}, X_{u_1}^{\rho_1})_{\theta,u} = X_u^{\rho}.$$

2. EXAMPLES

In this chapter we introduce some important examples of approximation spaces.

2.1. *Sequence Spaces*

As is usual, l_p with $0 < p \leqslant \infty$ stands for the quasi-Banach space of all p-summable scalar sequences $x = (\xi_m)$. Let f_n denote the set of all scalar sequences $a = (\alpha_m)$ possessing at most n coordinates $\alpha_m \neq 0$. We also consider the subset o_n which consists of all scalar sequences such that $\alpha_m = 0$ if $m > n$.

It follows easily that (l_p, f_n) and (l_p, o_n) are approximation schemes. We put

$$s_{p,u}^{\rho} := (l_p, f_n)_u^{\rho} \qquad \text{and} \qquad b_{p,u}^{\rho} := (l_p, o_n)_u^{\rho}.$$

Later on we shall show that $s_{p,u}^{\rho}$ coincides with the well-known Lorentz sequence space $l_{r,u}$, where $1/r = \rho + 1/p$. The sequence spaces $b_{p,u}^{\rho}$ can be considered as the discrete counterpart of the Besov function spaces which will be defined in the next section; cf. also [19].

2.2. *Function Spaces*

As is usual, L_p with $0 < p \leqslant \infty$ stands for the quasi-Banach space of all p-integrable scalar functions f defined on the unit interval.

Let F_n denote the set of all trigonometrical polynomials

$$a(s) = \alpha_1 + \sum_{m=1}^{\infty} [\alpha_{2m} \sin 2\pi ms + \alpha_{2m+1} \cos 2\pi ms]$$

possessing at most n coefficients $\alpha_m \neq 0$. We also consider the subset O_n which consists of all trigonometrical polynomials such that $\alpha_m = 0$ if $m > n$. It easily turns out that (L_p, F_n) and (L_p, O_n) are approximation schemes. We put

$$S^{\rho}_{p,u} := (L_p, F_n)^{\rho}_u \qquad \text{and} \qquad B^{\rho}_{p,u} := (L_p, O_n)^{\rho}_u.$$

A detailed theory of $S^{\rho}_{p,u}$ spaces has not been developed until now. However, there is an important result of Stečkin [20] which states that $S^{1/2}_{2,1}$ consists precisely of those periodic functions having an absolutely convergent Fourier series. On the other hand, it can be seen from approximation theory that $B^{\rho}_{p,u}$ are the famous Besov function spaces; cf. [2, 3, 6, 13, 21].

Note. In order to simplify our notation we will not indicate by an additional symbol that all spaces under consideration consist of periodic functions.

The above definitions extend to the case of X-valued functions, where X is a quasi-Banach space. Then we denote by $[L_p, X]$ the quasi-Banach space of all measurable and absolutely p-integrable X-valued functions $\underline{f}$ defined on the unit interval; cf. [22]. If the sets $[F_n, X]$ and $[O_n, X]$ are introduced canonically, we obtain the approximation spaces

$$[S^{\rho}_{p,u}, X] := ([L_p, X], [F_n, X])^{\rho}_u,$$

and

$$[B^{\rho}_{p,u}, X] := ([L_p, X], [O_n, X])^{\rho}_u.$$

2.3. *Operator Spaces*

Let $\mathfrak{L}(E, F)$ denote the Banach space of all (bounded linear) operators acting from the Banach space E into the Banach space F. An operator $S \in \mathfrak{L}(E, F)$ is called absolutely p-summing $(0 < p < \infty)$ if there exists a constant $\sigma \geqslant 0$ such that

$$\left(\sum_{i=1}^{n} \|Sx_i\|_F^p\right)^{1/p} \leqslant \sigma \sup \left\{ \left(\sum_{i=1}^{n} |\langle x_i, a\rangle|^p\right)^{1/p} : \|a\|_{E'} \leqslant 1 \right\}$$

for all finite systems of elements $x_1, \ldots, x_n \in E$. We put

$$\|S\|_{\mathscr{P}_p} =: \inf \sigma.$$

The quasi-Banach space of all absolutely p-summing operators from E into F will be denoted by $\mathfrak{P}_p(E, F)$.

It is convenient to define $\mathfrak{P}_\infty(E, F) := \mathfrak{L}(E, F)$, and $\|S\|_{\mathscr{P}_\infty} := \|S\|_{\mathscr{L}}$. Let $\mathfrak{F}_n(E, F)$ be the set of all operators $A \in \mathfrak{L}(E, F)$ having an image $M(A) := \{Ax: x \in E\}$ which is at most n-dimensional. It easily turns out that $(\mathfrak{P}_p(E, F), \mathfrak{F}_n(E, F))$ is an approximation scheme. We put

$$\mathfrak{P}^\rho_{p,u}(E, F) := (\mathfrak{P}_p(E, F), \mathfrak{F}_n(E, F))^\rho_u.$$

In the special case $p = \infty$ we also use the notation $\mathfrak{L}^\rho_u(E, F)$.

3. Theorems

In the sequel we prove some fundamental theorems.

3.1. *Representation Theorem*

First of all we establish the famous Representation Theorem which, in the case of function spaces, goes back to Besov [3] and Amanov [1]. The analogous result for operator spaces is due to Pietsch [16, 17]. We also mention that a very general theorem of this type was recently proved by Brudnij and Krugljak [4, 5]. See also the Equivalence Theorem in [6].

REPRESENTATION THEOREM. *Let (X, A_n) be an approximation scheme. Then $f \in X$ belongs to X^ρ_u if and only if there exist $a_k \in A_{2^k}$ such that*

$$f = \sum_{k=0}^{\infty} a_k \qquad \text{and} \qquad (2^{k\rho} \|a_k\|_X) \in l_u.$$

Moreover,

$$\|f\|^{\text{rep}}_{X^\rho_u} := \inf \|(2^{k\rho} \|a_k\|_X)\|_{l_u},$$

where the infimum is taken over all possible representations, defines an equivalent quasi-norm on X^ρ_u.

Necessity. Let $f \in X^\rho_u$. Then we choose $a^*_k \in A_{2^k-1}$ such that

$$\|f - a^*_k\|_X \leqslant 2\alpha_{2^k}(f, X).$$

Put $a_0 := o$, $a_1 := o$, and $a_{k+2} := a^*_{k+1} - a^*_k$ for $k = 0, 1, \ldots$. Obviously we have $a_k \in A_{2^k}$, and

$$f = \lim_k a^*_k = \sum_{k=0}^{\infty} a_k.$$

Moreover, it follows from

$$\|a_{k+2}\|_X \leqslant c_X[\|f-a_{k+1}^*\|_X + \|f-a_k^*\|_X] \leqslant 4c_X\alpha_{2^k}(f,X)$$

that $(2^{k\rho}\|a_k\|_X) \in l_u$, and we obtain the estimate

$$\|f\|_{X_u^\rho}^{\mathrm{rep}} \leqslant 2^{2\rho+2}c_X\|f\|_{X_u^\rho}^{\mathrm{exp}}.$$

Sufficiency. Without loss of generality we may suppose that $\|\cdot\|_X$ is a p-norm with $0<p<u$. If $f\in X$ can be written in the form $f=\sum_{k=0}^\infty a_k$ such that $a_k \in A_{2^k}$ and $(2^{k\rho}\|a_k\|_X)\in l_u$, then it follows from $\sum_{k=0}^{h-1} a_k \in A_{2^h-1}$ that

$$\alpha_{2^h}(f,X)^p \leqslant \left\| f - \sum_{k=0}^{h-1} a_k \right\|_X^p \leqslant \sum_{k=h}^\infty \|a_k\|_X^p.$$

In the case $0<u<\infty$ we put $q := u/p$, and choose α such that $\rho p > \alpha > 0$. Then

$$\begin{aligned}
\sum_{h=0}^\infty [2^{h\rho}\alpha_{2^h}(f,X)]^u &\leqslant \sum_{h=0}^\infty 2^{h\rho u}\left(\sum_{k=h}^\infty 2^{-k\alpha}2^{k\alpha}\|a_k\|_X^p\right)^q \\
&\leqslant \sum_{h=0}^\infty 2^{h\rho u}\left(\sum_{k=h}^\infty 2^{-k\alpha q'}\right)^{q/q'}\left(\sum_{k=h}^\infty 2^{k\alpha q}\|a_k\|_X^u\right) \\
&\leqslant c_1 \sum_{h=0}^\infty 2^{h(\rho u-\alpha q)}\sum_{k=h}^\infty 2^{k\alpha q}\|a_k\|_X^u \\
&\leqslant c_1 \sum_{k=0}^\infty 2^{k\alpha q}\|a_k\|_X^u \sum_{h=1}^k 2^{h(\rho u-\alpha q)} \\
&\leqslant c_2 \sum_{k=0}^\infty [2^{k\rho}\|a_k\|_X]^u < \infty.
\end{aligned}$$

This yields

$$\|f\|_{X_u^\rho}^{\mathrm{exp}} \leqslant c\,\|(2^{k\rho}\|a_k\|_X)\|_{l_u}.$$

Therefore we have $f\in X_u^\rho$ and $\|f\|_{X_u^\rho}^{\mathrm{exp}} \leqslant c\,\|f\|_{X_u^\rho}^{\mathrm{rep}}$. The case $u=\infty$ can be treated analogously.

As an immediate consequence of the Representation Theorem we get

PROPOSITION 5. *If $0<u<\infty$, then the linear subset A is dense in X_u^ρ.*

Proof. It easily turns out that every series $f=\sum_{k=0}^\infty a_k$ with $a_k\in A_{2^k}$ and $(2^{k\rho}\|a_k\|_X)\in l_u$ is even convergent relative to the quasi-norm $\|\cdot\|_{X_u^\rho}$.

An approximation scheme (X, A_n) is called *linear* if there exists a uniformly bounded sequence of linear projections P_n mapping X onto A_n.

Then it follows that

$$\|f - P_{n-1}f\|_X \leqslant c\alpha_n(f, X)$$

for all $f \in X$ and $n = 1, 2, \ldots$, where $c := c_X[1 + \sup \|P_n\|_{\mathscr{L}}]$.

Remark. In this case Butzer and Scherer [6] speak of an idempotent approximation procedure.

With the help of the projections

$$Q_k := P_{2^{k+1}-1} - P_{2^k-1}$$

we can formulate the

LINEAR REPRESENTATION THEOREM. *Let (X, A_n) be a linear approximation scheme. Then $f \in X$ belongs to X_u^ρ if and only if*

$$(2^{k\rho} \|Q_k f\|_X) \in l_u.$$

In this case we have

$$f = \sum_{k=0}^{\infty} Q_k f.$$

Moreover,

$$\|f\|_{X_u^\rho}^{\text{lin}} := \|(2^{k\rho} \|Q_k f\|_X)\|_{l_u}$$

is an equivalent quasi-norm on X_u^ρ.

EXAMPLE 1. The approximation scheme (l_p, o_n) is linear, since the sequence of the canonical projections has the required property.

EXAMPLE 2. The approximation scheme (L_p, O_n) with $1 < p < \infty$ is linear, since the sequence of the Fourier projections has the required property. The linearity fails in the limit cases $p = 1$ and $p = \infty$. For $0 < p < 1$ there even does not exist any bounded linear projection from L_p onto O_n.

3.2. *Reiteration Theorem*

For every approximation scheme (X, A_n) we have $A_n \subseteq X_u^\rho$. Therefore it follows that (X_u^ρ, A_n) is an approximation scheme, as well. This means that we can reiterate the construction of approximation spaces.

LEMMA 1. *There exists a constant $c > 0$ such that*

$$n^\rho \alpha_{2n-1}(f, X) \leqslant c\alpha_n(f, X_u^\rho) \qquad \textit{for all} \quad f \in X_u^\rho \textit{ and } n = 1, 2, \ldots.$$

Proof. We choose $a_1, a_2 \in A_{n-1}$ satisfying the estimates

$$\|f - a_1\|_{X_u^\rho} \leqslant 2\alpha_n(f, X_u^\rho) \text{ and } \|f - a_1 - a_2\|_X \leqslant 2\alpha_n(f - a_1, X).$$

Then

$$\begin{aligned}\alpha_{2n-1}(f, X) &\leqslant \|f - (a_1 + a_2)\|_X \leqslant 2\alpha_n(f - a_1, X) \leqslant 2n^{-\rho}\|f - a_1\|_{X_\infty^\rho} \\ &\leqslant c_0 n^{-\rho}\|f - a_1\|_{X_u^\rho} \leqslant 2c_0 n^{-\rho}\alpha_n(f, X_u^\rho).\end{aligned}$$

LEMMA 2. *There exists a constant $c > 0$ such that*

$$\|a\|_{X_u^\rho} \leqslant cn^\rho \|a\|_X \qquad \textit{for all} \quad a \in A_n \textit{ and } n = 1, 2, \ldots .$$

Proof. We consider the case where $0 < u < \infty$. Then

$$\begin{aligned}\|a\|_{X_u^\rho} &= \left(\sum_{k=1}^{n} [k^{\rho - 1/u}\alpha_k(a, X)]^u\right)^{1/u} \\ &\leqslant \left(\sum_{k=1}^{n} k^{\rho u - 1}\right)^{1/u} \|a\|_X \leqslant cn^\rho \|a\|_X.\end{aligned}$$

Remark. Within the framework of [6, Definition und Lemma 2.3.1] the results of Lemmas 1 and 2 are said to be a Jackson inequality and a Bernstein inequality, respectively.

Now we are in a position to establish the basic theorem of this section. Similar results are well known within the context of interpolation spaces, cf. [2, 3.5.3], and [21, 1.10.2]. See also [6, Korollar 2.3.2].

REITERATION THEOREM. *Let (X, A_n) be an approximation scheme. Then $(X_u^\rho)_v^\sigma = X_v^{\rho+\sigma}$.*

Proof. If $f \in (X_u^\rho)_v^\sigma$, we have $(2^{k\sigma}\alpha_{2^k}(f, X_u^\rho)) \in l_v$. Lemma 1 yields $(2^{k(\rho+\sigma)}\alpha_{2^k}(f, X)) \in l_v$. Hence $f \in X_v^{\rho+\sigma}$. This proves that $(X_u^\rho)_v^\sigma \subseteq X_v^{\rho+\sigma}$. Now let $f \in X_v^{\rho+\sigma}$, and consider a representation $f = \sum_{k=0}^\infty a_k$ such that $a_k \in A_{2^k}$ and $(2^{k(\rho+\sigma)}\|a_k\|_X) \in l_v$. Then we obtain from Lemma 2 that $(2^{k\sigma}\|a_k\|_{X_u^\rho}) \in l_v$. Consequently $f \in (X_u^\rho)_v^\sigma$. So we have $X_v^{\rho+\sigma} \subseteq (X_u^\rho)_v^\sigma$.

EXAMPLE 1. Since $(\alpha_n(x, l_\infty))$ is the non-increasing rearrangement of $x \in l_\infty$, we get $s_{\infty,u}^\rho = l_{r,u}$ with $1/r = \rho$, where $l_{r,u}$ is the well-known Lorentz sequence space. Now the Reiteration Theorem yields

$$s_{p,u}^\rho = (s_{\infty,p}^{1/p})_u^\rho = s_{\infty,u}^{1/p+\rho} = l_{r,u} \qquad \text{with} \quad 1/r = \rho + 1/p.$$

3.3. *Transformation Theorem*

The next result is very simple. However, it often happens in mathematics that trivial theorems can be extremely powerful.

TRANSFORMATION THEOREM. *Let (X, A_m) and (Y, B_n) be approximation schemes. Let $T \in \mathfrak{L}(X, Y)$. If there are constants $\lambda > 0$ and $c > 0$ such that*

$$T(A_m) \subseteq B_n \qquad \text{whenever} \qquad n \geqslant cm^\lambda,$$

then $T(X_u^{\lambda\rho}) \subseteq Y_u^\rho$.

Proof. We consider the case $\lambda \geqslant 1$ and put

$$N_m := \{n\colon c(m-1)^\lambda + 1 \leqslant n < cm^\lambda + 1\} \qquad \text{for} \quad m = 1, 2, \ldots .$$

Then

$$\operatorname{card}(N_m) \leqslant c_1 m^{\lambda - 1},$$
$$n^{\rho - 1/u} \leqslant c_2 m^{\lambda(\rho - 1/u)} \qquad \text{for} \quad n \in N_m,$$

and

$$\alpha_n(Tf, Y) \leqslant \|T\|_{\mathscr{L}}\, \alpha_m(f, X) \qquad \text{for} \quad f \in X \text{ and } n \in N_m.$$

Therefore, if $0 < u < \infty$, we obtain

$$\|Tf\|_{Y_u^\rho} = \left(\sum_{m=1}^{\infty} \sum_{N_m} [n^{\rho - 1/u} \alpha_n(Tf, Y)]^u\right)^{1/u}$$
$$\leqslant \left(\sum_{m=1}^{\infty} c_1 m^{\lambda - 1} [c_2 m^{\lambda(\rho - 1/u)} \|T\|_{\mathscr{L}}\, \alpha_m(f, X)]^u\right)^{1/u} \leqslant c\, \|f\|_{X_u^\rho}.$$

This proves the assertion for $\lambda \geqslant 1$ and $0 < u < \infty$. The remaining cases can be treated similarly.

Many applications of the Transformation Theorem will be given in the next chapter.

3.4. *Embedding Theorem*

In the sequel we consider a couple of quasi-Banach spaces X and Y which are continuously embedded into some linear topological Hausdorff space. Furthermore, let (X, A_n) and (Y, A_n) be approximation schemes built from the same sequence of subsets.

LEMMA 3. *Let $0 < \rho < \sigma < \infty$ and $0 < u \leqslant \infty$. Suppose that Y can be v-normed with $0 < v \leqslant 1$. Then the following conditions are equivalent:*

(1) *There exists a constant* $c_0 > 0$ *such that*

$$\|a\|_Y \leqslant c_0 n^\sigma \|a\|_X \qquad \textit{for all } a \in A_n \textit{ and } n = 1, 2, \ldots.$$

(2) *There exists a constant* $c_{\rho,u} > 0$ *such that*

$$\|a\|_Y \leqslant c_{\rho,u} n^{\sigma-\rho} \|a\|_{X_u^\rho} \qquad \textit{for all } a \in A_n \textit{ and } n = 1, 2, \ldots.$$

(3) *There exists a constant* $c_{\sigma,v} > 0$ *such that*

$$\|a\|_Y \leqslant c_{\sigma,v} \|a\|_{X_v^\sigma} \qquad \textit{for all } a \in A.$$

Proof. The implication $(2) \Rightarrow (1)$ follows immediately from Lemma 2. Because of $X_v^\sigma = (X_u^\rho)_v^{\sigma-\rho}$ and Lemma 2 we have

$$\|a\|_{X_v^\sigma} \leqslant cn^{\sigma-\rho} \|a\|_{X_u^\rho}$$

for all $a \in A_n$ and $n = 1, 2, \ldots$. Therefore $(3) \Rightarrow (2)$. By hypothesis we may suppose that $\|\cdot\|_Y$ is a v-norm. Then, given $a \in A$, there is a representation $a = \sum_{k=0}^\infty a_k$ such that $a_k \in A_{2^k}$, and $\|(2^{k\sigma} \|a_k\|_X)\|_{l_v} \leqslant 2 \|a\|_{X_v^\sigma}^{\text{rep}}$. Assuming (1) we now obtain

$$\|a\|_Y \leqslant \left\{ \sum_{k=0}^\infty \|a_k\|_Y^v \right\}^{1/v} \leqslant 2c_0 \|a\|_{X_v^\sigma}^{\text{rep}} \leqslant c_{\sigma,v} \|a\|_{X_v^\sigma}.$$

Hence $(1) \Rightarrow (3)$.

Now we are ready to establish the famous

EMBEDDING THEOREM. *Let X and Y be quasi-Banach spaces which are continuously embedded into some linear topological Hausdorff space. Furthermore, let (X, A_n) and (Y, A_n) be approximation schemes built from the same sequence of subsets. Suppose that there exist constants $\sigma > 0$ and $c > 0$ such that*

$$\|a\|_Y \leqslant cn^\sigma \|a\|_X \qquad \textit{for all } a \in A_n \textit{ and } n = 1, 2, \ldots.$$

Then $X_u^{\rho+\sigma} \subseteq Y_u^\rho$. In particular, if Y can be v-normed with $0 < v \leqslant 1$, we have $X_v^\sigma \subseteq Y$.

Proof. Since the linear subset A is dense in X_v^σ, condition (3) of the preceding lemma yields $X_v^\sigma \subseteq Y$. Now it follows from the Reiteration Theorem that $X_u^{\rho+\sigma} \subseteq (X_v^\sigma)_u^\rho \subseteq Y_u^\rho$.

EXAMPLE 1. Let $0 < q < p \leqslant \infty$. Because of Hölder's inequality we have

$$\|a\|_{l_q} \leqslant n^{1/q-1/p} \|a\|_{l_p} \qquad \text{for all } a \in f_n \text{ and } n = 1, 2, \ldots.$$

This yields the embeddings

$$s^{\rho}_{p,u} \subseteq s^{\sigma}_{q,u} \qquad \text{if} \quad \rho + 1/p = \sigma + 1/q.$$

From the formula $l_{r,u} = s^{\rho}_{p,u}$ with $1/r = \rho + 1/p$ we see that even identity holds.

Analogously we get

$$b^{\rho}_{p,u} \subseteq b^{\sigma}_{q,u} \qquad \text{if} \quad \rho + 1/p = \sigma + 1/q.$$

EXAMPLE 2. Let $1 \leqslant p < q \leqslant \infty$. Then Nikolskij's inequality [13, 3.4.3] states that

$$\|a\|_{L_q} \leqslant c n^{1/p - 1/q} \|a\|_{L_p} \qquad \text{for all } a \in O_n \text{ and } n = 1, 2, \ldots .$$

This yields the embeddings

$$B^{\rho}_{p,u} \subseteq B^{\sigma}_{q,u} \qquad \text{if} \quad \rho - 1/p = \sigma - 1/q.$$

Nessel and Wilmes [12] observed that the above estimate remains true also for $a \in F_n$ if $1 \leqslant p \leqslant 2$. There are, however, counterexamples for $2 < p < \infty$ and $q = \infty$. The case $2 < p < q < \infty$ seems to be open.

EXAMPLE 3. Let $1 \leqslant q \leqslant 2 \leqslant p \leqslant \infty$. Then the Carl–Lewis inequality [7, 11] states that

$$\|A\|_{\mathscr{P}_q} \leqslant c n^{1/q - 1/p} \|A\|_{\mathscr{P}_p} \qquad \text{for all } A \in \mathfrak{F}_n(E, F) \text{ and } n = 1, 2, \ldots .$$

This yields the embeddings

$$\mathfrak{P}^{\rho}_{p,u}(E, F) \subseteq \mathfrak{P}^{\sigma}_{q,u}(E, F) \qquad \text{if} \quad \rho + 1/p = \sigma + 1/q.$$

In particular, we get the inclusion $\mathfrak{L}^{1/2}_1 \subseteq \mathfrak{P}_2$, which was first proved by König [9].

Remark. Another result of this type is due to the author [15, 18.6.3]. Let $\mathfrak{N}(E, F)$ denote the Banach space of all nuclear operators $T \in \mathfrak{L}(E, F)$ equipped with the norm $\|\cdot\|_{\mathscr{N}}$. Then

$$\|A\|_{\mathscr{N}} \leqslant n \|A\|_{\mathscr{L}} \qquad \text{for all } A \in \mathfrak{F}_n(E, F) \text{ and } n = 1, 2, \ldots,$$

and it follows that $\mathfrak{L}^1_1(E, F) \subseteq \mathfrak{N}(E, F)$. Furthermore, it turns out that many inclusions proved in the theory of operator ideals can be understood as embeddings within the context of approximation spaces.

3.5. *Composition Theorem*

Now we prove a so-called

COMPOSITION THEOREM. *Let (X, A_n), (Y, B_n), and (Z, C_n) be approximation schemes. If M is a bounded bilinear map from $X \times Y$ into Z such that $M(A_n, Y) \subseteq C_n$ and $M(X, B_n) \subseteq C_n$ for $n = 1, 2, \ldots$, then*

$$M(X_u^\rho, Y_v^\sigma) \subseteq Z_w^{\rho+\sigma} \qquad \textit{whenever} \quad 1/u + 1/v = 1/w.$$

Proof. The assertion follows immediately from

$$\alpha_{m+n-1}(M(f, g), Z) \leqslant \|M\| \, \alpha_m(f, X) \, \alpha_n(g, Y),$$

and Hölder's inequality.

EXAMPLE 1. Let $x \in l_p$ and $y \in l_q$. If $M(x, y)$ is defined to be the coordinatewise product $x \cdot y = (\xi_m \eta_m)$ of the sequences $x = (\xi_m)$ and $y = (\eta_m)$, then Hölder's inequality yields $M(x, y) \in l_r$ with $1/r = 1/p + 1/q$. Consequently we have

$$s_{p,u}^\rho \cdot s_{q,v}^\sigma \subseteq s_{r,w}^{\rho+\sigma} \qquad \text{and} \qquad b_{p,u}^\rho \cdot b_{q,v}^\sigma \subseteq b_{r,w}^{\rho+\sigma}.$$

EXAMPLE 2. Let $f \in L_p$ and $g \in L_q$ such that $1/p + 1/q \geqslant 1$. If $M(f, g)$ is defined to be the convolution

$$f * g(s) = \int_0^1 f(t) \, g(s-t) \, dt$$

of the functions f and g (periodically extended on the real line), then it is well known that $M(f, g) \in L_r$ with $1/r = 1/p + 1/q - 1$; cf. [23, Chap. II,1]. Consequently we have

$$S_{p,u}^\rho * S_{q,v}^\sigma \subseteq S_{r,w}^{\rho+\sigma} \qquad \text{and} \qquad B_{p,u}^\rho * B_{q,v}^\sigma \subseteq B_{r,w}^{\rho+\sigma}.$$

EXAMPLE 3. Let $T \in \mathfrak{P}_p(E, F)$ and $S \in \mathfrak{P}_q(F, G)$ such that $1/p + 1/q \leqslant 1$. If $M(S, T)$ is defined to be the product ST of the operators S and T, then it is well-known that $ST \in \mathfrak{P}_r(E, G)$ with $1/r = 1/p + 1/q$; cf. [15, 20.2.4]. Consequently we get

$$\mathfrak{P}_{q,v}^\sigma(F, G) \circ \mathfrak{P}_{p,u}^\rho(E, F) \subseteq \mathfrak{P}_{r,w}^{\rho+\sigma}(E, G).$$

3.6. *Commutation Theorem*

Let (X, A_n) be an approximation scheme, and consider the quasi-Banach space $[L_p, X_u^\rho]$. On the other hand, if $[L_p, A_n]$ consists of all A_n-valued functions $\mathbf{f} \in [L_p, X]$, we get an approximation scheme $([L_p, X], [L_p, A_n])$. The corresponding approximation spaces will be denoted by $[L_p, X]_u^\rho$.

COMMUTATION THEOREM. *There holds*

(1) $[L_p, X_u^\rho] \subseteq [L_p, X]_u^\rho$ *if* $0 < p \leqslant u \leqslant \infty$

and

(2) $[L_p, X_u^\rho] \supseteq [L_p, X]_u^\rho$ *if* $0 < u \leqslant p \leqslant \infty$ *and* $u \neq \infty$.

Proof. (1) We consider a measurable step-function $\underline{\mathbf{f}} = \sum_{h=1}^m f_h \chi_h$, where $f_1, ..., f_m \in X_u^\rho$, and $\chi_1, ..., \chi_m$ are characteristic functions of pairewise disjoint measurable subsets $\Omega_1, ..., \Omega_m$. Choose $a_{hn} \in A_{n-1}$ such that $\|f_h - a_{hn}\|_X \leqslant 2\alpha_n(f_h, X)$, and put

$$\underline{\mathbf{a}}_n := \sum_{h=1}^m a_{hn}\chi_h.$$

Let $0 < p \leqslant u < \infty$. Since $\underline{\mathbf{a}}_n \in [L_p, A_{n-1}]$, it follows that

$$\begin{aligned}\alpha_n(\underline{\mathbf{f}}, [L_p, X]) &\leqslant \|\underline{\mathbf{f}} - \underline{\mathbf{a}}_n\|_{[L_p,X]} = \left(\sum_{h=1}^m \|f_h - a_{hn}\|_X^p \mu_h\right)^{1/p} \\ &\leqslant 2\left(\sum_{h=1}^m \alpha_n(f_h, X)^p \mu_h\right)^{1/p}.\end{aligned}$$

Here μ_h denotes the Lebesgue measure of Ω_h. Therefore we get

$$\begin{aligned}\|\underline{\mathbf{f}}\|_{[L_p,X]_u^\rho} &= \left\{\sum_{n=1}^\infty [n^{\rho-1/u}\alpha_n(\underline{\mathbf{f}}, [L_p, X])]^u\right\}^{1/u} \\ &\leqslant 2\left\{\sum_{n=1}^\infty \left[n^{\rho-1/u}\left(\sum_{h=1}^m \alpha_n(f_h, X)^p \mu_h\right)^{1/p}\right]^u\right\}^{1/u} \\ &\leqslant 2\left\{\sum_{h=1}^m \left(\sum_{n=1}^\infty [n^{\rho-1/u}\alpha_n(f_h, X)]^u\right)^{p/u} \mu_h\right\}^{1/p} \\ &\leqslant 2\left\{\sum_{h=1}^m \|f_h\|_{X_u^\rho}^p \mu_h\right\}^{1/p} = 2\,\|\underline{\mathbf{f}}\|_{[L_p,X_u^\rho]}.\end{aligned}$$

So we have

$$\|\underline{\mathbf{f}}\|_{[L_p,X]_u^\rho} \leqslant 2\,\|\underline{\mathbf{f}}\|_{[L_p,X_u^\rho]}$$

for all X_u^ρ-valued measurable step-functions $\underline{\mathbf{f}}$. Now the assertion follows from the fact that these functions are dense in $[L_p, X_u^\rho]$.

(2) Let $\underline{\mathbf{f}} \in [L_p, X]_u^\rho$. Then there exists a representation $\underline{\mathbf{f}} = \sum_{k=0}^\infty \underline{\mathbf{a}}_k$ such that $\underline{\mathbf{a}}_k \in [L_p, A_{2^k}]$ and $(2^{k\rho}\|\underline{\mathbf{a}}_k\|_{[L_p,X]}) \in l_u$. It can be shown that $\underline{\mathbf{f}}(s) = \sum_{k=0}^\infty \underline{\mathbf{a}}_k(s)$ almost everywhere. So, if $0 < u \leqslant p < \infty$, we obtain

$$\|\underline{\mathbf{f}}(s)\|_{X_u^\rho}^{\text{rep}} \leqslant \left\{\sum_{k=0}^\infty [2^{k\rho}\,\|\underline{\mathbf{a}}_k(s)\|_X]^u\right\}^{1/u},$$

and therefore

$$\left\{\int_0^1 (\|\underline{\mathbf{f}}(s)\|_{X_u^\rho}^{\text{rep}})^p \, ds\right\}^{1/p} \leqslant \left\{\int_0^1 \left(\sum_{k=0}^{\infty} [2^{k\rho} \|\underline{\mathbf{a}}_k(s)\|_X]^u\right)^{p/u} ds\right\}^{1/p}$$

$$\leqslant \left\{\sum_{k=0}^{\infty} \left[2^{k\rho} \left(\int_0^1 \|\underline{\mathbf{a}}_k(s)\|_X^p \, ds\right)^{1/p}\right]^u\right\}^{1/u} < \infty.$$

This implies that $\underline{\mathbf{f}} \in [L_p, X_u^\rho]$.

Remark. If $u = \infty$, then we get into trouble with respect to the measurability of $\underline{\mathbf{f}}$ as an X_u^ρ-valued function.

Remark. An analog of the above Commutation Theorem is well known within the framework of interpolation spaces; cf. [2, 5.8.6] and [21, 1.18.4].

4. Applications

Now we apply the theory of approximation spaces in order to get results about the distribution of Fourier coefficients and eigenvalues.

4.1. *Fourier Coefficients*

For every scalar function $f \in L_1$ the sequence of Fourier coefficients is defined by

$$\xi_1(f) := \int_0^1 f(s) \, ds,$$

$$\xi_{2m}(f) := 2 \int_0^1 f(s) \sin 2\pi ms \, ds,$$

and

$$\xi_{2m+1}(f) := 2 \int_0^1 f(s) \cos 2\pi ms \, ds.$$

There are many classical results concerning the problem how the asymptotic behaviour of $(\xi_m(f))$ depends on certain properties of the function f; cf. [18]. All these facts are summarized in the following theorems.

THEOREM 1. *Let* $1/r = \rho + 1/2$. *Then the assertions* $f \in S_{2,u}^\rho$ *and* $(\xi_m(f)) \in l_{r,u}$ *are equivalent.*

Proof. According to the Fischer–Riesz theorem $Tf := (\xi_m(f))$ defines an isomorphism between L_2 and l_2. Hence the Transformation Theorem tells us that T is also an isomorphism between $S^\rho_{2,u}$ and $s^\rho_{2,u}$. But the latter space coincides with $l_{r,u}$.

If $1 \leqslant p \leqslant \infty$, then p' denotes the conjugate exponent defined by $1/p + 1/p' = 1$, and we put $p^+ := \max(2, p')$.

THEOREM 2. *Let* $1/r = \rho + 1/p^+$. *Then* $f \in S^\rho_{p,u}$ *implies* $(\xi_m(f)) \in l_{r,u}$.

Proof. In the case $1 \leqslant p \leqslant 2$ we see from the Hausdorff–Young theorem that $Tf := (\xi_m(f))$ defines an operator from L_p into $l_{p'}$. Now the assertion follows from the Transformation Theorem. If $2 \leqslant p \leqslant \infty$, then $S^\rho_{p,u} \subseteq S^\rho_{2,u}$ yields the conclusion.

Because of $B^\rho_{p,u} \subseteq S^\rho_{p,u}$ we also have

THEOREM 3. *Let* $1/r = \rho + 1/p^+$. *Then* $f \in B^\rho_{p,u}$ *implies* $(\xi_m(f)) \in l_{r,u}$.

Remark. Using examples constructed in [23, Chap. V,2 + 4 and VI,3] we can see that the preceding results are best possible.

4.2. *Integral Operators*

Let K be a scalar kernel defined on the unit square. By $\underline{\mathbf{k}}(s)$ we denote the function $K(s, \cdot)$ where s is fixed. Then $\underline{\mathbf{k}}: s \to \underline{\mathbf{k}}(s)$ is an abstract function. We say that the kernel K belongs to $[B^\rho_{p,u}, B^\sigma_{q,v}]$ if this is true for the $B^\sigma_{q,v}$-valued function $\underline{\mathbf{k}}$. Kernels of type $[L_p, L_q]$, $[L_p, B^\sigma_{q,v}]$, and $[B^\rho_{p,u}, L_q]$ are defined analogously.

In the following we investigate integral operators

$$S_K: g(t) \to f(s) = \int_0^1 K(s, t)\, g(t)\, dt$$

generated by a kernel K. A first result is well-known from the theory of absolutely p-summing operators; cf. [8].

LEMMA 4. *Let* $1/p + 1/q \leqslant 1$. *Then*

(0) $K \in [L_p, L_q] \Rightarrow S_K \in \mathfrak{P}_{q'}(L_{q'}, L_{q'})$.

Now we are in a position to establish

THEOREM 4. *Let* $1/p + 1/q \leqslant 1$. *Then the following implications are true*:

(1) $K \in [L_p, B^\sigma_{q,v}], \qquad p \leqslant v \qquad \Rightarrow \qquad S_K \in \mathfrak{P}^\sigma_{q',v}(L_{q'}, L_{q'})$,

(2) $K \in [B^{\rho}_{p,u}, L_q] \quad \Rightarrow \quad S_K \in \mathfrak{P}^{\rho}_{q',u}(L_{q'}, L_{q'})$,

(3) $K \in [B^{\rho}_{p,u}, B^{\sigma}_{q,v}] \quad \Rightarrow \quad S_K \in \mathfrak{P}^{\rho+\sigma}_{q',u}(L_{q'}, L_{q'})$.

Proof. We consider the operator T transforming every kernel K into the corresponding integral operator S_K. Then the Transformation Theorem, the Commutation Theorem, and Lemma 4 yield

(1) $T([L_p, B^{\sigma}_{q,v}]) \subseteq T([L_p, L_q]^{\sigma}_v) \subseteq \mathfrak{P}^{\sigma}_{q',v}(L_{q'}, L_{q'})$.

Applying once again the Transformation Theorem to (0) and (1), we obtain (2) and (3), respectively.

4.3. *Convolution Operators*

Let f be a 1-periodic scalar function defined on the real line. Then we put

$$K_f(s, t) := f(s - t).$$

The well-known theorem about the continuity of the shift operator yields

LEMMA 5. *Let* $0 < q < \infty$. *Then*

(0) $f \in L_q \quad \Rightarrow \quad K_f \in [L_\infty, L_q]$.

Now we get

THEOREM 5. *Let* $0 < q < \infty$. *The following implications are true*:

(1) $f \in B^{\sigma}_{q,v}, \quad v \neq \infty \quad \Rightarrow \quad K_f \in [L_\infty, B^{\sigma}_{q,v}]$,

(2) $f \in B^{\rho+\sigma}_{q,u} \quad \Rightarrow \quad K_f \in [B^{\rho}_{\infty,u}, B^{\sigma}_{q,v}]$.

Proof. We consider the operator T transforming every scalar function f into the corresponding convolution kernel K_f. Then the Transformation Theorem, the Commutation Theorem, and Lemma 5 yield

(1) $T(B^{\sigma}_{q,v}) \subseteq [L_\infty, L_q]^{\sigma}_v \subseteq [L_\infty, B^{\sigma}_{q,v}]$.

Applying once again the Transformation Theorem to (1) we obtain (2).

4.4. *Eigenvalues of Operators*

In the sequel let E be a complex Banach space. Since every operator $S \in \mathfrak{P}^{\rho}_{p,u}(E, E)$ can be approximated by operators of finite rank, it follows from Riesz's theory that S possesses a set of eigenvalues which is at most countable. Let $(\lambda_n(S))$ denote the *sequence of* these *eigenvalues* counted according to their multiplicities and ordered such that $|\lambda_1(S)| \geqslant |\lambda_2(S)| \geqslant \cdots \geqslant 0$. If S has less than n eigenvalues, then we put $\lambda_n(S) = 0$. Now we state an interesting generalization of Weyl's theorem which goes back to Carl and König [10].

THEOREM 6. *Let* $2 \leqslant p \leqslant \infty$ *and* $1/r = \rho + 1/p$. *Then* $S \in \mathfrak{P}^{\rho}_{p,u}(E, F)$ *implies* $(\lambda_n(S)) \in l_{r,u}$.

Proof. First we treat the case where $S \in \mathfrak{P}^{\rho}_{2,u}(E, E)$. Then there exists a representation $S = \sum_{k=0}^{\infty} S_k$ such that $S_k \in \mathfrak{F}_{2^k}(E, E)$ and $(2^{k\rho} \|S_k\|_{\mathscr{P}_2}) \in l_u$. As shown in the theory of absolutely 2-summing operators [15, 17.3.7], we can find factorizations $S_k = X_k A_k$ with $A_k \in \mathfrak{L}(E, l_2^{2^k})$ and $X_k \in \mathfrak{L}(l_2^{2^k}, E)$ such that

$$\|A_k\|^2_{\mathscr{P}_2} = \|X_k\|^2_{\mathscr{L}} = \|S_k\|_{\mathscr{P}_2}.$$

Consider l_2 as the orthogonal sum of the spaces $l_2^{2^k}$ with $k = 0, 1, \ldots$. Let $J_k \in \mathfrak{L}(l_2^{2^k}, l_2)$ and $Q_k \in \mathfrak{L}(l_2, l_2^{2^k})$ denote the canonical injections and surjections, respectively. Then it follows from $(2^{k\rho} \|A_k\|^2_{\mathscr{P}_2}) \in l_u$ and $(2^{k\rho} \|X_k\|^2_{\mathscr{L}}) \in l_u$ that

$$A := \sum_{k=0}^{\infty} J_k A_k \in \mathfrak{P}^{\rho/2}_{2,2u}(E, l_2) \qquad \text{and} \qquad X := \sum_{k=0}^{\infty} X_k Q_k \in \mathfrak{L}^{\rho/2}_{2u}(l_2, E).$$

Since $S = XA$ is related to $T = AX$, we have $\lambda_n(S) = \lambda_n(T)$; cf. [15, 27.3.3]. Moreover, using $\mathfrak{P}_2(l_2, l_2) = \mathfrak{L}^{1/2}_2(l_2, l_2)$, the Reiteration Theorem, and the Composition Theorem, we get

$$T \in \mathfrak{P}^{\rho/2}_{2,2u}(E, l_2) \circ \mathfrak{L}^{\rho/2}_{2u}(l_2, E) = \mathfrak{P}^{\rho}_{2,u}(l_2, l_2) = \mathfrak{L}^{\rho + 1/2}_{u}(l_2, l_2).$$

Hence the classical Weyl theorem yields $(\lambda_n(T)) \in l_{r,u}$ with $1/r = \rho + 1/2$. This proves the assertion for $p = 2$.

Now we suppose that $2 < p \leqslant \infty$ and $\rho + 1/p > 1/2$. Then it follows from the Embedding Theorem that $\mathfrak{P}^{\rho}_{p,u}(E, E) \subseteq \mathfrak{P}^{\sigma}_{2,u}(E, E)$, where $\sigma := \rho + 1/p - 1/2$. Consequently, for $S \in \mathfrak{P}^{\rho}_{p,u}(E, E)$ we get $(\lambda_n(S)) \in l_{r,u}$ with $1/r = \sigma + 1/2 = \rho + 1/p$.

Unfortunately the above method does not work for $\rho + 1/p \leqslant 1/2$. However, in this case we can apply König's interpolation procedure as presented in [8, 9].

Remark. If $p = \infty$ and $\rho \leqslant 1/2$, then it is possible to choose a natural number m such that $m\rho > 1/2$. Now we obtain from $S \in \mathfrak{L}^{\rho}_{u}(E, E)$ that $S^m \in \mathfrak{L}^{m\rho}_{u/m}(E, E)$, and therefore $(\lambda_n(S^m)) \in l_{r/m, u/m}$ with $1/r = \rho$. Hence, by the Spectral Mapping Theorem, it follows that $(\lambda_n(S)) \in l_{r,u}$.

Summarizing Theorems 4 and 6 we get some interesting results about the distribution of eigenvalues of integral operators.

THEOREM 7. *Let $1/p + 1/q \leqslant 1$. The following implications are true*:

(1) $K \in [L_p, B^{\sigma}_{q,v}], p \leqslant v \;\Rightarrow\; (\lambda_n(S_K)) \in l_{r,v}$ *with* $1/r = \sigma + 1/q^+$,

(2) $K \in [B^{\rho}_{p,u}, L_q] \;\Rightarrow\; (\lambda_n(S_K)) \in l_{r,u}$ *with* $1/r = \rho + 1/q^+$,

(3) $K \in [B^{\rho}_{p,u}, B^{\sigma}_{q,v}] \;\Rightarrow\; (\lambda_n(S_K)) \in l_{r,u}$ *with* $1/r = \rho + \sigma + 1/q^+$.

Remark. Using another method the author [19] was able to show that the condition $1/p + 1/q \leqslant 1$ can be replaced by the weaker assumptions $1/p + 1/q < \rho + \sigma + 1$ and $1 \leqslant p, q \leqslant \infty$.

Finally we recall that the (complex) Fourier coefficients

$$\lambda_1(f) := \int_0^1 f(s)\, ds,$$

$$\lambda_{2m}(f) := \int_0^1 f(s) \exp(-2\pi i m s)\, ds,$$

$$\lambda_{2m+1}(f) := \int_0^1 f(s) \exp(2\pi i m s)\, ds,$$

coincide with the eigenvalues of the convolution operator generated by the kernel K_f. Moreover, we have

$$\lambda_1(f) = \xi_1(f), \qquad \lambda_{2m+1}(f) + \lambda_{2m}(f) = \xi_{2m+1}(f),$$
$$\lambda_{2m+1}(f) - \lambda_{2m}(f) = i\xi_{2m}(f).$$

Therefore, summarizing Theorems 5 and 7 we once again get the statement of Theorem 3.

Remark. The counterexamples constructed in the theory of trigonometrical series show that the results stated in Theorem 7 are best possible.

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